Negative Autoregulation using Hill Functions

$$\frac{dX}{dt} = f(X) - \alpha X \qquad \text{with} \quad f(X) = \beta \frac{K^n}{\underbrace{K^n + X^n}_{\text{Repressed}}}$$

Assume strong repression $\Rightarrow \left(\frac{X}{K}\right)^n \gg 1$ high binding $\Rightarrow$ low K value

$$\frac{dX}{dt} = \beta \frac{K^n}{X^n} - \alpha X \qquad u(t) = X(t)^{n+1}$$

$$\hookrightarrow \frac{du}{dt} = (n+1) \cdot X^n \cdot \frac{dX}{dt}$$

$$\Rightarrow \frac{du}{dt} = (n+1) \cdot X^n \cdot \left(\beta \frac{K^n}{X^n} - \alpha X\right)$$

$$= (n+1) \cdot \beta K^n - (n+1)\alpha \cdot u$$

$$\Rightarrow \frac{du}{dt} = a - b \cdot u$$

$$\hookrightarrow u(t) = u_{ST}(1 - \exp(-bt))$$

$$\Rightarrow u(t) = u_{ST}(1 - \exp(-(n+1)\alpha t)),$$

$$X(t) = X_{ST} \cdot \left[1 - \exp(-(n+1)\alpha t)\right]^{\frac{1}{n+1}} \Rightarrow \boxed{T_{1/2} = \frac{1}{(n+1)\alpha} \cdot \ln\left(\frac{2^{n+1}}{2^{n+1} - 1}\right)}$$

Compare $T_{1/2}(n)$ with $T_{1/2}(n=0) \Rightarrow$ $n=1$, 0.2
$n=2$, 0.06
$n=3$, 0.02